Hyodo-Kato Cohomology of the First Covering of Drinfeld Spaces

ZHENGHUI LI

Abstract. Let K be a finite extension of $\mathbb{Q}_p$. We verify that in the middle degree of the Hyodo-Kato cohomology of the first covering of the Drinfeld space, one can realize the Jacquet-Langlands and local Langlands correspondence for depth-zero supercuspidal representations.

1. Intoduction

Let K be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ be the ring of integers, ϖ be a uniformizer and $\mathbb{F}_q$ be the residue field. Fix a positive integer d. Choose an algebraic closure $\overline{K}$ of K and let $C := \widehat{\overline{K}}$ be the completion of the algebraic closure of $\overline{K}$. Let $G = \mathrm{GL}_{d+1}(K)$, and let D be the $(d+1)^2$-dimensional central division algebra over K with invariant $\frac{1}{d+1}$. Denote by $D^\times$ be the group of units of D and $\mathcal{O}_D$ be the ring of integers. Drinfeld (cf. [9],[10]) introduced the p-adic symmetric space $\mathbb{H}_K^d$ and its coverings Σ_K^n (See Section 2.3 for recall).

On one hand, Drinfeld and Carayol ([1]) conjectured a decomposition of the limit of the l-adic cohomology of the tower Σ_C^n, predicting the space realizes of certain Jacquet-Langlands and local Langlands correspondence. This conjecture has been extensively studied over the past decades.

On the other hand, the Drinfeld tower has also played key role in recent geometric appoach to p-adic Langlands program of $\mathrm{GL}_2(\mathbb{Q}_p)$. In [7, Sec. 5], the authors use global methods to show that the Hyodo-Kato cohomology of the Drinfeld tower realizes Jacquet-Langlands and local Langlands correspondence (for supercuspidal representations) in the case $d = 1$. In fact, the Hyodo-Kato cohomology is expected to serve as the 'p-adic companion' of the l-adic cohomology (cf. [8, Sec. 8]), thus a similar realization is expected for higher dimensional cases.

In this note, we consider the first covering of the Drinfeld space and verify the expectation for the depth-zero supercuspidal representations.

Let θ be a primitive character of $\mathbb{F}_{q^{d+1}}^\times$. Let $\rho(\theta)$ be the depth-zero supercuspidal representation of $D^\times$ associated to θ (See Section 4.1), $\pi(\theta)$ be the supercuspidal representation of G under Jacquet-Langlands correspondence, $\sigma(\theta)$ be the Weil-Deligne representation given by the local Langlands correspondence and $M(\theta)$ be a (φ, N, G_K)-module whose associated Weil-Deligne representation is $\sigma(\theta)$.

Theorem 1.1 (Theorem 4.3)**.** *We have an isomorphism of G-modules compatible with action of φ, N and G_K.*

$$\mathrm{Hom}_{D^\times}(\rho(\theta), H^r_{\mathrm{HK},c}(\mathcal{M}^{1,\varpi}_{Dr,C})) \cong \begin{cases} \pi(\theta)\otimes M(\theta) & r = d \\ 0 & others. \end{cases} \tag{1.1}$$

Junger Damian has realized Jacquet-Langlands correspondence on $H^r_{\mathrm{dR},c}(\mathcal{M}^{1,\varpi}_{Dr,C})$. We prove the result by using Hyodo-Kato isomorphism and computing several operators on the Hyodo-Kato cohomology.

Acknowledgement. I would like to thank Prof. Wieslawa Niziol for pointing out this expectation to me and helpful discussions. I also thank Junger Damian and Gabriel Dospinescu for helpful communications.

2. Drinfeld Space and First Covering

2.1. **Bruhat-Tit Tree.** We recall the Bruhat-Tit building and the reduction map on the Drinfeld Space. Details can be found in [5, Sec. 1, 6].

For an integer $d \geq 0$ and a $(d+1)$-dimensional vector space V_K over K, we let $\mathcal{BT}$ be the Bruhat-Tit tree (of $\mathrm{PGL}(V_K)$) with a natural G-action. It is the following simplicial complex: the vertices $\mathcal{BT}_0$ contains dilation classes of lattices $s = [L]$ in V_K and a k-cell is a $(k+1)$-tuple $\{s_0, ..., s_k\}$ such that, after permuting s_i, we can find representatives $s_i = [L_i]$ together with a flag (with strict inclusion)

$$L_0 \supset L_1 \supset L_2 \supset ... \supset L_k \supset \pi L_0.$$

The type of such a k-cell σ is a sequence of numbers $(e_0, ..., e_k)$ where $e_i := \dim_{\mathbb{F}_q} L_i/L_{i+1}$. For simplices τ, σ, we denote $\tau \leq \sigma$ if τ is a face of σ.

Let $|\mathcal{BT}|$ be the topological realization of $\mathcal{BT}$. We denote the interior of a cell σ by $\mathring{\sigma} := \sigma \backslash (\cup_{\sigma' \subsetneq \sigma} \sigma')$. Given a vertex $v_0 \in \mathcal{T}$, the star $st(v_0)$ is the union of $\mathring{\sigma}$ such that $v_0 \in \sigma$ and $\overline{st(v_0)}$ is the closure. For a simplex τ, we define

$$st(\tau) := \bigcup_{\tau \leq \sigma} \mathring{\sigma} = \bigcap_{v \in \tau} st(v)$$

$$\overline{st(\tau)} := \bigcap_{v \in \tau} \overline{st(v)}.$$

By a classical theorem of Goldman and Iwahori ([12]), there is a bijection between $|\mathcal{BT}|$ and dilation classes of real norms on V_K.

2.2. **Drinfeld Space.** Now we let $V_K = K^{d+1}$ thus indentify $\mathbb{P}(V_K)$ with $\mathbb{P}^d_K$. Let $\mathbb{H}^d_K$ be the d-dimensional Drinfeld half-space over K. It is the K-rigid subvariety of $\mathbb{P}^d_K$ that is the complement of all K-rational hyperplanes (cf. [9]). There is a G-equivariant reduction map

$$\tau : \mathbb{H}^d_K(C) \to |\mathcal{BT}|$$

via mapping $x = [x_0, ..., x_d] \in \mathbb{H}^d_K(C)$ to the dilation class of norms l_x on V_K, where

$$l_x(v) = |\sum_{i=0}^{d} x_i v_i|.$$

For a simplex σ, we define $\mathbb{H}^d_{K,\sigma} := \tau^{-1}(\sigma)$ and similarly for $\mathbb{H}^d_{K,\mathring{\sigma}}$ and $\mathbb{H}^d_{K,st(\sigma)}$.

We refer the following facts to [23, Sec. 2.1] and [16, Sec. 3].

There is a semistable model $\mathbb{H}^d_{\mathcal{O}_K}$ of $\mathbb{H}^d_K$ due to Deligne via gluing local models. The moduli interpretation of the local models $\mathbb{H}^d_{\mathcal{O}_K,\sigma}$ (for a simplex σ of $\mathcal{BT}$) of $\mathbb{H}^d_{K,\sigma}$ could be found in [21, Appendix to Sec. 3] or [23, Sec. 2.1] and $\mathbb{H}^d_{\mathcal{O}_K} := \mathrm{colim}_{\sigma \in \mathcal{BT}} \mathbb{H}^d_{\mathcal{O}_K,\sigma}$. Therefore, the special fiber also admits such a decomposition $\mathbb{H}^d_{\mathbb{F}_q} = \cup_\sigma \mathbb{H}^d_{\mathbb{F}_q,\sigma}$ satisfying $\mathbb{H}^d_{K,\sigma} =]\mathbb{H}^d_{\mathbb{F}_q,\sigma}[$ and we can define $\mathbb{H}^d_{\mathbb{F}_q,\mathring{\sigma}}$, $\mathbb{H}^d_{\mathbb{F}_q,st(\sigma)}$ by taking complements and unions. (We warn that $\mathbb{H}^d_{\mathbb{F}_q}$ is the special fiber of $\mathbb{H}^d_{\mathcal{O}_K}$ rather than $\mathbb{P}^d_{\mathbb{F}_q} \setminus \cup_{H \in \mathcal{H}} H$)

The irreducible components of the special fiber $\mathbb{H}^d_{\mathbb{F}_q}$ are parametrized by vertices of $\mathcal{BT}$ and are isomorphic projective smooth varieties. For $s \in \mathcal{BT}_0$, the corresponding irreducible component is $\mathbb{H}^d_{\mathbb{F}_q,st(s)}$ which is isomorphic to the variety obtained from $\mathbb{P}^d_{\mathbb{F}_q}$ by first blowing up all $\mathbb{F}_q$-rational points, then blowing up strict transformation of all

$\mathbb{F}_q$-rational line, etc. Thus the inclusion of s into $st(s)$ identifies $\mathbb{H}^d_{\mathbb{F}_q,s}$ with $\mathbb{P}^d_{\mathbb{F}_q}\backslash \cup_{H\in\mathcal{H}} H$ where $\mathcal{H}$ is the set of rational hyperplanes of $\mathbb{P}^d_{\mathbb{F}_q}$. Then we get an admissible open covering of $\mathbb{H}^d_K$ by $(\mathbb{H}^d_{K,st(s)})_{s\in\mathcal{BT}_0} = (]\mathbb{H}^d_{\mathbb{F}_q,st(s)}[)_{s\in\mathcal{BT}_0}$ and intersections of such opens are of the form $\mathbb{H}^d_{K,st(\sigma)}$ for some simplex σ in $\mathcal{BT}$.

There is an explicit description of the spaces $\mathbb{H}^d_{K,\mathring{\sigma}}$ as the following (cf. [5, Sec. 6.4]): Assume the simplex σ with type $(e_0,...,e_k)$ is represented by

$$L_0 \supset L_1 \supset L_2 \supset ... \supset L_k \supset \pi L_0.$$

Let $d_i = e_i + \cdots + e_k$. Consider the affinoid subdomain of $\mathbb{B}^r_K$

$$C_r := \{(x_1,...,x_r) \in \mathbb{B}^r_K : \ \forall a \in \mathcal{O}_K^{r+1}\backslash \pi\mathcal{O}_K^{r+1}, \ |\langle (1,x), a\rangle| = 1\}$$

and the 'multiannulus'

$$A_k := \{(t_1,...,t_k) \in \mathbb{B}^k_K : \ |\pi| < |t_k| < \cdots < |t_1| < 1\}.$$

By choosing a basis x_i of V_K adapted to σ, the morphisms

$$\mathbb{H}^d_{K,\mathring{\sigma}} \to C_{e_i-1} : [x_0,...,x_d] \mapsto (\frac{x_{d_{i-1}+1}}{x_{d_{i-1}}}, \frac{x_{d_{i-1}+2}}{x_{d_{i-1}}}, ..., \frac{x_{d_i-1}}{x_{d_{i-1}}})$$

$$\mathbb{H}^d_{K,\mathring{\sigma}} \to A_k : [x_0,...,x_d] \mapsto (\frac{x_{d_0}}{x_{d_k}}, \frac{x_{d_1}}{x_{d_k}}, ..., \frac{x_{d_{k-1}}}{x_{d_k}})$$

induces an isomorphism

$$\mathbb{H}^d_{K,\mathring{\sigma}} \xrightarrow{\cong} \prod_{i=0}^{k} C_{e_i-1} \times A_k =: C_\sigma \times A_k. \tag{2.1}$$

2.3. **The First Covering of Drinfeld Space.**

2.3.1. *Moduli Interpretation.* We recall the construction of coverings of the Drinfeld half-space following [10]. Let D be the central division algebra over K of dimension $(d+1)^2$ with invariant $1/(d+1)$. Let $\mathcal{O}_D$ be its ring of integers and Π_D be a uniformizer.

Definition 2.1. Let B be an $\mathcal{O}_K$-algebra and $\mathcal{O}_{d+1}$ be the ring of integers of a maximal unramified extension of K in D.

(1) An formal $\mathcal{O}_K$-module over B is a formal group over B together with an action of $\mathcal{O}_K$ such that it induces the natural action on the tangent space.
(2) A formal $\mathcal{O}_D$-module over B is a formal $\mathcal{O}_K$-module together with an $\mathcal{O}_D$-action which extends the action of $\mathcal{O}_K$. A formal $\mathcal{O}_D$-module X is special if $\mathrm{Lie}(X)$ is a $\mathcal{O}_{d+1} \otimes_{\mathcal{O}_K} B$-module locally free of rank 1.

We fix a special formal $\mathcal{O}_D$-module Φ over $k = \overline{\mathbb{F}}_q$. Let $\mathcal{O}_{\breve{K}} := W(k)$ and $\breve{K} := \mathcal{O}_{\breve{K}}[1/p]$. By [22, Lem 3.60], any such two special formal $\mathcal{O}_D$-modules are isogenous. Let $\mathrm{Nilp}_{\mathcal{O}_K}$ be the category of $\mathcal{O}_K$-algebras such that image of ϖ is nilpotent. We define a functor $\mathcal{G} : \mathrm{Nilp}_{\mathcal{O}_K} \to \mathrm{Sets}$ which maps $B \in \mathrm{Nilp}_{\mathcal{O}_K}$ to isomorphic classes of triples (ψ, X, ρ) where

(1) $\psi : k \to B/\varpi$ is an $\mathbb{F}_q$-homomorphism.
(2) X is a special formal $\mathcal{O}_D$-module over A of height $(d+1)^2$.
(3) $\rho : \Phi \otimes_k B/\varpi B \dashrightarrow X_{B/\varpi B}$ is a quasi-isogeny of height zero.

Theorem 2.2 (Drinfeld)**.** *The functor $\mathcal{G}$ is represented by $\mathbb{H}^d_{\mathcal{O}_{\breve{K}}}$.*

One could also form another functor $\widetilde{\mathcal{G}}:\mathrm{Nilp}_{\mathcal{O}_K}\to$ Sets which maps $B\in\mathrm{Nilp}_{\mathcal{O}_K}$ to isomorphic classes of triples $(\psi,X,\widetilde{\rho})$ where ψ,X are the same as above and $\widetilde{\rho}$ is a quasi-isogeny between $\Phi\otimes_k B/\varpi B$ and $X_{B/\varpi B}$, without restriction on the height. Then the functor $\widetilde{\mathcal{G}}$ has a decomposition

$$\widetilde{\mathcal{G}}=\coprod_{h\in\mathbb{Z}}\mathcal{G}^{(h)}$$

where $\mathcal{G}^{(h)}$ corresponds to those $(\psi,X,\widetilde{\rho})$ such that $\widetilde{\rho}$ has height $(d+1)h$. Each $\mathcal{G}^{(h)}$ is non-canonically isomorphic to $\mathcal{G}$. By [10, Sec. 2] and [22, Thm. 3.72], the functor $\widetilde{\mathcal{G}}$ is represented by a formal scheme $\widehat{\mathcal{M}}^0_{Dr}$ and the decomposition above induces a non-canonical isomorphism

$$\widehat{\mathcal{M}}^0_{Dr}\cong\mathbb{H}^d_{\mathcal{O}_{\breve{K}}}\times\mathbb{Z}.$$

The space $\widehat{\mathcal{M}}^0_{Dr}$ admits a Weil descent datum relative to $\breve{K}/K$ which is given by the composition of the canonical Weil descent datum on $\mathbb{H}^d_{\mathcal{O}_{\breve{K}}}$ and translate by 1 on $\mathbb{Z}$.

Let $\mathfrak{X}$ (resp. $\widetilde{\mathfrak{X}}$) be the universal formal special $\mathcal{O}_D$-module over $\mathbb{H}^d_{\mathcal{O}_{\breve{K}}}$ (resp. $\widehat{\mathcal{M}}^0_{Dr}$). For $n\geq 0$, multiplication by Π^n_D induces an isogeny on $\mathfrak{X}$ (resp. $\widetilde{\mathfrak{X}}$) and the group $\mathfrak{X}[\Pi^n_D]:=\ker(\mathfrak{X}\xrightarrow{\Pi^n_D}\mathfrak{X})$ (resp. $\widetilde{\mathfrak{X}}[\Pi^n_D]$) is finite flat of rank $p^{(d+1)n}$ over $\mathbb{H}^d_{\mathcal{O}_{\breve{K}}}$ (resp. $\widehat{\mathcal{M}}^0_{Dr}$). We set $\Sigma^0:=\mathbb{H}^d_{\breve{K}}$ and $\mathcal{M}^0_{Dr}:=(\widehat{\mathcal{M}}^0_{Dr})^{rig}\cong\Sigma^0\times\mathbb{Z}$. For $n\geq 1$, we define

$$\Sigma^n:=\mathfrak{X}[\Pi^n_D]\backslash\mathfrak{X}[\Pi^{n-1}_D];\ \mathcal{M}^n_{Dr}:=\widetilde{\mathfrak{X}}[\Pi^n_D]\backslash\widetilde{\mathfrak{X}}[\Pi^{n-1}_D].$$

The projection $\Sigma^n\to\Sigma^0$ and $\mathcal{M}^n_{Dr}\to\mathcal{M}^0_{Dr}$ are finite étale morphisms with Galois group $\mathcal{O}^\times_D/(1+\Pi^n_D\mathcal{O}_D)$. There exist an action of $G,D^\times$ and a Weil descent data on the tower $\{\mathcal{M}^n_{Dr}\}_n$ together with a $G\times D^\times$-equivariant period morphism $\xi:\mathcal{M}^0_{Dr}\to\mathbb{H}^d_K$ such that when we base change $\mathbb{H}^d_K$ to $\breve{K}$ the morphism is compatible with Weil descent data (the data on $\mathbb{H}^d_{\breve{K}}$ is via Galois descent) (See [4, Sec 3.1] for descriptions of actions).

2.3.2. *The First Covering.* In particular, the first covering $\pi:\Sigma^1\to\mathbb{H}^d_{\breve{K}}$ is finite étale with Galois group $\mathbb{F}^\times_{q^d}\cong\mathcal{O}^\times_D/(1+\Pi_D\mathcal{O}_D)$. Put $N=q^{d+1}-1$ and let r be the composition

$$r:\Sigma^1\xrightarrow{\pi}\mathbb{H}^d_{\breve{K}}\xrightarrow{\tau}|\mathcal{BT}|.$$

For a simplicial complex η of $\mathcal{BT}$, define $\Sigma^1_\eta:=r^{-1}|\eta|$. It is clear that $\Sigma^1_\eta=\pi^{-1}\mathbb{H}^d_{\breve{K},\eta}$.

Damien ([18, Thm. A]) proved the vanishing of analytic picard group of $\mathbb{H}^d_L$ for any complete extension L/K (Note that $\mathrm{Pic}_{\mathrm{an}}(\mathbb{H}^d_L)=\mathrm{Pic}_{\acute{e}t}(\mathbb{H}^d_L)$ cf. [11, Prop 8.2.3]). Apply classification of Raynaud scheme together with a study of invertible functions on $\mathbb{H}^d_L$ ([18, Thm 7.1]), he shows the following result:

Theorem 2.3 ([17, Thm. 4.9])**.** *There exists $u\in\mathcal{O}^\times(\mathbb{H}^d_{\breve{K}})$ such that*

$$\Sigma^1\cong\mathbb{H}^d_{\breve{K}}((\pi u)^{1/N}):=\underline{\mathrm{Spec}}_{\Sigma^0}(\mathcal{O}_{\Sigma^0}[X]/(X^N-\varpi u))\tag{2.2}$$

In fact, the element u has an explicit description and satisfies the following property. Let $s=[\mathcal{O}^{d+1}_K]$ be the standard lattice of V_K and fix $b\in(\mathbb{F}_q)^d\backslash\{0\}$. Put

$$u(z):=(-1)^d(b_0z_0+\cdots+b_dz_d)^{-N}\prod_{a\in(\mathbb{F}_q)^d\backslash\{0\}}(a_0z_0+\cdots+a_dz_d)$$

which is an invertible function over $\mathbb{H}^d_{\mathbb{F}_q,s}\cong\mathbb{P}^d\backslash\cup_{H\in\mathcal{H}}H$. In fact, the image of $u(z)$ in $\mathcal{O}^\times(\mathbb{H}^d_{\mathbb{F}_q,s})/(\mathcal{O}^\times(\mathbb{H}^d_{\mathbb{F}_q,s}))^N$ does not depend on b so we can choose $b=(0,...,0,1)$. We can lift u_1 to an invertible function on $\mathbb{H}^d_{\breve{K},s}$ and $\mathbb{H}^d_{\breve{K},st(s)}$ and still denote it by u_1. Then $u|_{\mathbb{H}^d_{\breve{K},st(s)}}\equiv u_1\pmod{\mathcal{O}^\times(\mathbb{H}^d_{\breve{K},st(s)})^N}$.

3. Cohomology of Σ^1_s and Deligne-Lusztig Variety

3.1. **Deligne Lusztig variety.** Recall the Deligne-Lusztig variety asssociated to $G = \mathrm{GL}_{d+1,\mathbb{F}_q}$ (cf. [6]) . Let B be the subgroup of upper-triangle matrices of G, U be the strict upper-triangle matrices and T be the diagonal matrices. Let $W = N_G(T)/T \cong S_{d+1}$ be the Weyl group identified with the permutation group. Assume w is a matrix corresponds to permutation $(0, 1, ..., d)$ and we regard $\overline{\mathbb{F}}_q$-points of the flag variety $X := G/B$ as the set of Borel subgroups of $G(\overline{\mathbb{F}}_q)$. Define $X(w) \subseteq X$ to be the subvariety whose $\overline{\mathbb{F}}_q$-points containing Borel subgroups B' such that B' and $F(B')$ are in relative position w, i.e

$$X(w) = \{gB : g^{-1}F(g) \in BwB\} \subseteq G/B.$$

There is a variety $\tilde{X}(w)$ above $X(w)$:

$$\widetilde{X}(w) := \{gU : g^{-1}F(g) \in UwU\} \subseteq G/U$$

with an obvious map $\pi : \widetilde{X}(w) \to X(w);\ gU \mapsto gB$. We have a commutative diagram

$$\begin{tikzcd} \widetilde{X}(w) \arrow[r, "i"] \arrow[d, "\pi"'] & G/U \arrow[d, "\pi"] \\ X(w) \arrow[r, "i"] & G/B \end{tikzcd}$$

By [6, Sec 2.2], the variety $X(w)$ can be identified with $\mathbb{H}^d_{\mathbb{F}_q}$. In fact, it is the non-vanishing locus of

$$\prod_{a\in\mathbb{P}^d(\mathbb{F}_q)} (a_0X_0 + \cdots + a_dX_d) = c\ \cdot\ \det((X_i^{q^j})_{0\le i,j\le d})$$

in $\mathbb{P}^d_{\mathbb{F}_q}$ where $c \in \mathbb{F}_q^\times$ depending on choice of representatives of a. In this case, the variety $\widetilde{X}(w)$ can be identified with the closed subvariety $\mathrm{DL}^d_{\mathbb{F}_q}$ of $\mathbb{A}^{d+1}_{\mathbb{F}_q}$ defined by equation

$$\det((X_i^{q^j})_{0\le i,j\le d})^{q-1} = (-1)^d$$

such that $\pi : \widetilde{X}(w) \to X(w)$ is induced by the natural map $\mathbb{A}^{d+1}_{\mathbb{F}_q}\backslash\{0\} \to \mathbb{P}^d_{\mathbb{F}_q}$.

Theorem 3.1 ([23, Thm. 2.5.4], [16, Lem. 6.3])**.** *Let s be a vertex of $\mathcal{BT}$ and $\breve{K}_N = \breve{K}(\varpi^{1/N})$. Then $\Sigma^1_{\breve{K}_N,s}$ admits a smooth model $\widehat{\Sigma}^1_{\mathcal{O}_{\breve{K}_N},s}$ such that the special fiber $\overline{\Sigma}^1_s$ is isomorphic to $\mathrm{DL}^d_{\overline{\mathbb{F}}_q}$. Moreover, the isomorphism is $\mathrm{GL}_{d+1}(\mathcal{O}_K) \times \mathbb{F}^\times_{q^{d+1}}$-equivariant.*

Let $\theta : \mathbb{F}^\times_{q^{d+1}} \to \breve{K}$ be a character. We call it primitive if does not factor through any norm $\mathrm{Norm} : \mathbb{F}^\times_{q^{d+1}} \to \mathbb{F}^\times_{q^s}$ for $s \le d$. For V a representation of $\mathbb{F}^\times_{q^{d+1}}$, we write $\mathrm{Hom}_{\mathbb{F}^\times_{q^{d+1}}}(\theta, V)$ as $V[\theta]$.

Proposition 3.2 ([16, Prop. 6.6])**.** *Assume σ is a simplex of dimension greater than zero, then $H^j_{\mathrm{dR},c}(\Sigma^1_{st(\sigma)})[\theta] = 0$ for all j and primitive θ.*

3.1.1. *G-action.* We have the following theorem

Theorem 3.3 ([14, Cor. 4.5])**.** *For any $l \neq p$, fix an isomorphism $\overline{\mathbb{Q}}_l \cong \overline{K}$ and let θ be a nonsingular character of $\mathbb{F}^\times_{q^{d+1}}$ then*

$$\overline{\pi}_\theta := H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/\overline{K})[\theta] := (H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/K) \otimes_K \overline{K})[\theta]$$

is isomorphic to

$$\overline{\pi}_{\theta,l} := H^d_{\acute{e}t,c}(\mathrm{DL}^d_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_l)[\theta]$$

as representations of $G(\mathbb{F}_q)$ *on a field of characteristic zero. Moreover, if* $i \neq d$ *then the* θ*-eigenspace of* $H^i_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/\overline{K})$ *is zero.*

Remark 3.4. By Deligne-Lusztig correspondence, the $G(\mathbb{F}_q)$-representation $\overline{\pi}_{\theta,l}$ is irreducible with dimension $(q-1)(q^2-1)\cdots(q^d-1)$, so is $\overline{\pi}_\theta$. Define

$$H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/K^{\mathrm{ur}}) := \mathrm{colim}_{K'\subseteq K^{\mathrm{ur}}} H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/K')$$

where K'/K_0 are finite unramified extensions. Then $\overline{\pi}_\theta$ is realizable over K^{ur} as $\overline{\pi}_\theta \cong H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/\check{K})[\theta] \otimes_{\check{K}} \overline{K}$. In fact, such a realization is unique (up to isomorphism) by applying the following well-known lemma and using irreducibility. we also use $\overline{\pi}_\theta$ to denote the representation $H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/K^{\mathrm{ur}})[\theta]$ or $H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_q}/\check{K})[\theta]$ if there is no confusion.

Lemma 3.5. *Let* G *be a finite group. Let* K *be a field in characteristic zero and* L/K *be an extension of* K*. Assume* V, V' *are two finite dimensional* K*-representations of* G*, then there is an isomorphism*

$$\mathrm{Hom}_G(V,V') \otimes_K L \xrightarrow{\cong} \mathrm{Hom}_G(V_L, V'_L).$$

Proof. It is clear there is an isomorphism

$$\mathrm{Hom}(V,V') \otimes_K L \xrightarrow{\cong} \mathrm{Hom}(V_L, V'_L).$$

For elements $g_i \in G$ $(1 \le i \le n)$, let $\rho(g_i)$ (resp. $\rho'(g_i)$) be their image in $\mathrm{End}(V)$ (resp. $\mathrm{End}(V')$), then we can identify

$$\mathrm{Hom}_G(V,V') \cong \ker(\mathrm{Hom}(V,V') \xrightarrow{\alpha_i} \oplus_{i=1}^n \mathrm{Hom}(V,V'))$$

where $\alpha_i(f) = \rho'(g_i)\circ f - f \circ \rho(g_i)$. □

3.1.2. *Frobenius.* Let σ be the lifting of the Frobenius of $\mathbb{F}_q$ to $W(\mathbb{F}_q)[1/p]$. Assume θ is a primitive character of $\mathbb{F}^\times_{q^{d+1}}$ and $h \in \mathbb{F}^\times_{q^{d+1}}$. Since $\mathbb{F}^\times_{q^{d+1}}$ is a cyclic group of order $(q^{d+1}-1)$ and $K_{d+1} := W(\mathbb{F}_{q^{d+1}})[1/p]$ contains $(q^{d+1}-1)$-th root of unit, the θ-action on $H^d_{\mathrm{rig},c}(\mathrm{DL}_{\mathbb{F}_q}/\check{K})$ descents to $H^d_{\mathrm{rig},c}(\mathrm{DL}_{\mathbb{F}_{q^{d+1}}}/K_{d+1})$. By [14, Remarks (1) before Lem 1.3], the space $H^d_{\mathrm{rig},c}(\mathrm{DL}_{\mathbb{F}_{q^{d+1}}}/K_{d+1})[\theta]$ is stable under the action of φ^{d+1}. Since the action of $G(\mathbb{F}_q)$ commutes with φ and rigid cohomology commutes with extension of base field, we can write

$$H^d_{\mathrm{rig},c}(\mathrm{DL}_{\mathbb{F}_{q^{d+1}}}/K_{d+1})[\theta] \cong M_\theta \otimes V_\theta$$

where V_θ is a $G(\mathbb{F}_q)$-representation of dimension $\prod_{i=1}^d (q^i-1)$ and M_θ is a 1-dimensional φ^{d+1}-module over K_{d+1}.

Proposition 3.6. φ^{d+1} *acts on* M_θ *via multiplying by* $(-1)^d q^{d(d+1)/2}$.

Proof. Note that φ^{d+1} is σ^{d+1}-semilinear which fixes $\mathbb{F}_{q^{d+1}}$, it suffices to see

$$\mathrm{Tr}(\varphi^{d+1}|H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_{q^{d+1}}})[\theta]) = (-1)^d q^{d(d+1)/2}\prod_{i=1}^d (q^i-1).$$

Use the same argument as in [15, Page 171] (replace Lefschetz trace formula of crystalline cohomology by rigid cohomology) and apply theorem 3.3, we get

$$(-1)^d \,\mathrm{Tr}(\varphi^{d+1}|H^d_{\mathrm{rig},c}(\mathrm{DL}^d_{\mathbb{F}_{q^{d+1}}})[\theta]) = \frac{1}{q^{d+1}-1}\sum_{h\in\mathbb{F}^\times_{q^{d+1}}} \theta(h)\#\,\mathrm{Fix}(\varphi^{d+1}h^{-1}).$$

To compute it, recall ([14, Sec 4]) the following expression of Deligne-Lusztig variety.

Let $z_0 = 1$ and $z_1, ..., z_d$ to be d variables. Set $\delta := \det((z_i^{q^j})_{0\leq i,j\leq d})$ and

$$\Pi := -\prod_{a\in\mathbb{F}_q^{d+1}-\{0\}}\sum_{i=0}^{d} a_i z_i.$$

Then $\Pi = (-1)^d\delta^{q-1}$. Set $A = \mathbb{F}_q[z_1, ..., z_d][\frac{1}{\Pi}]$, $B = A[X_0]/(1-X_0^{q^{d+1}-1}\Pi)$, $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. The map $\widetilde{X}(w) \to X(w)$ can be $G(\mathbb{F}_q)$-equivariantly identified with the natural map $X \to Y$. The $\mathbb{F}_{q^{d+1}}^\times$-action on $X_{\mathbb{F}_{q^{d+1}}} \to Y_{\mathbb{F}_{q^{d+1}}}$ can be written as $h(X_0, z_1, ..., z_d) = (hX_0, z_1, ..., z_d)$ where $h \in \mathbb{F}_{q^{d+1}}^\times$ and $(X_0, z_1, ..., z_d) \in X(k)$. Thus $\mathrm{Fix}(\varphi^{d+1}h^{-1})$ is the set $(X_0, z_1, ..., z_d) \in k^{d+1}$ such that

$$\begin{cases} X_0^{q^{d+1}} = hX_0 \\ z_i \in \mathbb{F}_{q^{d+1}} \\ X_0^{q^{d+1}-1}\Pi(z) = 1 \end{cases}$$

Since $X_0 \neq 0$, the third equation becomes $\Pi(z) = (-1)^d\delta^{q-1}(z) = h^{-1}$. Note that $z_i^{q^{d+1}} = z$, so $\delta^q = \det((z_{i,j}^{q^{j+1}})_{0\leq i,j\leq d}) = (-1)^d\delta$ which means $h = 1$. Now the claim follows from [6, Prop 2.3] that

$$\#Y(\mathbb{F}_{q^{d+1}}) = \prod_{i=1}^{d}(q^{d+1} - q^i) = q^{d(d+1)/2}\prod_{i=1}^{d}(q^i - 1).$$

$\square$

3.2. Hyodo-Kato cohomology of $\Sigma^1_{s,C}$.

3.2.1. *Overconvergent Hyodo-Kato cohomology.* We recall compactly supported overconvergent Hyodo-Kato cohomology in [20]. Let $L = K$ or C and X is a smooth rigid analytic variety over L. Then we have arithmetic (when $L = K$) or geometric (when $L = C$) Hyodo-Kato cohomology $R\Gamma_{\mathrm{HK}}(X)$ of X (See [2, Sec 4.2, 4.3]) and completed geometric (when $L = C$) Hyodo-Kato cohomology $R\Gamma_{\mathrm{HK},\breve{F}}(X)$ of X (See [3, Sec. 4]). The definitions can be moved to condensed maths and we get solid version of these cohomologies. Note that if there exists a semistable model of X, we have local global compatibility of corresponding cohomologies (cf. [2, Prop. 4.11, 4.23], [3, Lem. 4.2]).

In overconvergent setting, there are two definitions: one is via Hyodo-Kato cohomology of Gross-Klonne ([2]), the other one is locally via presentation of dagger affinoid, use the geometric rigid analytic Hyodo-Kato cohomology and then descent. These two constructions gives the same cohomology (cf. [3, Lem. 4.14]). In particular, we have the following local-global compatibility: Assume $\mathscr{X}/\mathcal{O}_F$ is a semistable weak formal scheme where F is a finite extension of K. Let $X^\dagger := \mathscr{X}_{\mathcal{O}_C,\eta}$ be the generic fiber of $\mathscr{X}_{\mathcal{O}_C}$ which is a smooth dagger variety. Then we have a quasi-isomorphism of (φ, N)-modules

$$R\Gamma_{\mathrm{HK}}(X^\dagger) \simeq R\Gamma_{\mathrm{rig}}^{\log}(\mathscr{X}_1^0/L_0) \otimes_{L_0}^{\square} K^{ur}$$

$$R\Gamma_{\mathrm{HK},\breve{F}}(X^\dagger) \simeq R\Gamma_{\mathrm{rig}}^{\log}(\mathscr{X}_1'^0/\breve{K})$$

If $\mathscr{X}$ is quasi-compact and quasi-separated, then the cohomology groups above are finite dimensional and the solidification is not necessary.

For compactly supported overconvergent setting. We first have a local definition: For a smooth dagger affinoid variety $X^\dagger$ over L with presentation $\{X_h\}_{h\in\mathbb{N}}$, the corresponding (local) overconvergent Hyodo-Kato cohomology is defined via

$$R\Gamma^\natural_{\mathrm{HK},*}(X^\dagger) := \mathrm{colim}_h\, R\Gamma_{\mathrm{HK},*}(X_h)$$

where $* \in \{\emptyset, \check{F}\}$ when $L = C$. The compactly supported cohomologies are defined by

$$R\Gamma^{\natural}_{\mathrm{HK},*,c}(X^{\dagger}) := [R\Gamma^{\natural}_{\mathrm{HK},*}(X^{\dagger}) \to R\Gamma^{\natural}_{\mathrm{HK},*}(\partial X^{\dagger})]$$

where the cohomology of boundary is defined as

$$R\Gamma^{\natural}_{\mathrm{HK},*}(\partial X^{\dagger}) := \mathrm{colim}_h \, R\Gamma_{\mathrm{HK},*}(X_h \backslash \widehat{X^{\dagger}}).$$

For general smooth dagger variety, the corresponding compactly supported Hyodo-Kato cohomology $R\Gamma_{\mathrm{HK},*,c}(X^{\dagger})$ is defined via analytic descent (cf. [20, Sec. 4.1]). When $L = C$ and X a smooth dagger affinoid over L, we have local-global compactibility

$$R\Gamma_{\mathrm{HK},*,c}(X^{\dagger}) \simeq R\Gamma^{\natural}_{\mathrm{HK},*,c}(X^{\dagger}).$$

We have the Hyodo-Kato duality for smooth dagger affinoid variety:

Theorem 3.7 ([20, Sec. 6.5.3])**.** *For a smooth dagger affinoid variety $X^{\dagger}$ over K pure of dimension d, there is a perfect pairing compatible with (φ, N, G_K) actions*

$$H^i_{\mathrm{HK},*}(X^{\dagger}_C) \times H^{2d-i}_{\mathrm{HK},*,c}(X^{\dagger}_C)\{d\} \to L_K$$

where $L_F = K^{ur}$ when $* = \emptyset$ and $L_F = \check{K}$ when $* = \check{F}$. Moreover, such a pairing is compatible with the pairing of de Rham cohomology under the Hyodo-Kato map.

3.2.2. *Cohomology of $\Sigma^1_{s,C}$.*

Lemma 3.8. *For a primative character θ, we have*

$$H^i_{\mathrm{HK},c}(\Sigma^1_{s,C})[\theta] \cong \begin{cases} \mathcal{M}_{\theta} \otimes \overline{\pi}_{\theta} & i = d \\ 0 & i \neq d \end{cases}$$

where $\mathcal{M}_{\theta}$ is a (φ^{d+1}, N, G_K)-module of rank 1. Here φ^{d+1} acts as multiplying by $(-1)^d q^{d(d+1)/2}$, the action of the monodromy operator N is trivial, the G_K-action factors through $\mathrm{Gal}(\check{K}_N/\check{K}) \cong \mathbb{F}^{\times}_{q^{d+1}}$ via the character θ.

Proof. By functoriality of the Hyodo-Kato cohomology, the action of $G(\mathbb{F}_q) \times \mathbb{F}^{\times}_{q^{d+1}}$ commutes with the action of φ, N and G_K. By theorem 2.3, $\Sigma^1_{s,C}$ admits a smooth formal model whose special fiber is the Deligne-Lusztig variety. Note that $\Sigma^1_{s,C}$ is an (smooth) affinoid domain of $\mathbb{B}^d_C$ (See section 2.2) which admits a dagger structure and we view it as a dagger affinoid space. Since the Deligne-Lusztig variety admits a smooth weak formal model, the natural map (where 0 means log structure $1 \to 0$)

$$H^i_{\mathrm{rig}}(\mathrm{DL}_{\mathbb{F}_{q^{d+1}}}/K_{n+1}) \to H^i_{\mathrm{rig}}(\mathrm{DL}^0_{\mathbb{F}_{q^{d+1}}}/K^0_{n+1})$$

induces an isomorphism of φ-modules which is equivariant under the action of $\mathbb{F}^{\times}_{q^{d+1}}$ and $G(\mathbb{F}_q)$. Using duality of Hyodo-Kato cohomology (Theorem 3.7) and rigid cohomology ([19, Thm. 1.2.3]) and the local-global compatibility, we can apply previous results on rigid cohomology of Deligne-Lusztig varieties to Hyodo-Kato cohomology of $\Sigma^1_{s,C}$. Thus we get the decomposition, the dimension of $\mathcal{M}_{\theta}$ is one and φ^{d+1} acts as multiplying by $(-1)^d q^{d(d+1)/2}$. The action of the monodromy operator N is trivial because $N\varphi = q\varphi N$. We need to determine the Galois action, note that there is a formal scheme $\Sigma^1_{s,\mathcal{O}_{K_N}}$ such that its special fiber is the rational Deligne-Lusztig $\mathrm{DL}_{\mathbb{F}_q}$ and $\Sigma^1_{s,\check{K}_N}$ is the base change of its generic fiber to $\check{K}_N$.

Lemma 3.9. *There is an isomorphism of finite dimensional G_{K_N}-modules*

$$H^s_{\mathrm{HK},c}(\Sigma^1_{s,K_N}) \otimes_{K_0} K^{ur} \cong H^s_{\mathrm{HK},c}(\Sigma^1_{s,C})$$

where the action on the left hand side is via the coefficient K^{ur}.

Proof. The non-completed Hyodo-Kato satisfies Galois descent (cf. [2, Prop. 4.26]), we know compactly supported Hyodo-Kato cohomology also satisfies Galois descent. Thus we have a map

$$H^s_{\mathrm{HK}}(\Sigma^1_{s,K_N}) \cong H^s_{\mathrm{HK}}(\Sigma^1_{s,C})^{G_{K_N}} \hookrightarrow H^s_{\mathrm{HK}}(\Sigma^1_{s,C}).$$

By local-global compatibility, there is an isomorphism $H^s_{\mathrm{HK}}(\Sigma^1_{s,K_N}) \cong H^s_{\mathrm{rig}}(\mathrm{DL}_{\mathbb{F}_q}/K_0)$. Thus, we have the following commutative diagram of finite dimensional G_{K_N}-modules

$$\begin{tikzcd}
H^s_{\mathrm{HK}}(\Sigma^1_{s,K_N}) \otimes K^{ur} \arrow[r] \arrow[d, "\cong"'] & H^s_{\mathrm{HK}}(\Sigma^1_{s,C}) \arrow[d, "\cong"] \\
H^s_{\mathrm{rig}}(\mathrm{DL}_{\mathbb{F}_q}/K_0) \otimes_{K_0} K^{\mathrm{ur}} \arrow[r, "="] & H^s_{\mathrm{rig}}(\mathrm{DL}_{\mathbb{F}_q}/K_0) \otimes_{K_0} K^{\mathrm{ur}}.
\end{tikzcd}$$

We can conclude from duality for Hyodo-Kato cohomology (Theorem 3.7), duality for rigid cohomology and local global compatibility of compactly supported Hyodo-Kato cohomology.

□

In particular, the $G_{\breve{K}_N}$-action on $H^s_{\mathrm{HK},c}(\Sigma^1_{s,\breve{K}})$ factors through $\mathrm{Gal}(\breve{K}_N/\breve{K})$. Note that, after choosing ϖ_N a N-th power root of ϖ, there is an isomorphism

$$\Sigma^1_{\breve{K}_N,s} \cong \mathrm{Sp}\,\mathcal{O}_{\mathbb{H}^d_{\breve{K}_N,s}}[X']/(X'^N - u)$$

where $X' = X/\varpi_N$. Then $\mathrm{Gal}(\breve{K}_N/\breve{K})$ can be identified with μ_N via $g \mapsto g(\pi_N)/\pi_N \in \mu_N$ and the Galois action of $\mathrm{Gal}(\breve{K}_N/\breve{K})$ is identified with the action of $\mathbb{F}^\times_{q^{d+1}}$ by choosing an appropriate isomorphism $\mu_N \cong \mathbb{F}^\times_{q^{d+1}}$. So the action of $G_{\breve{K}}$ on $H^d_{\mathrm{HK},c}(\Sigma^1_{s,C})[\theta]$ factors through $\mathrm{Gal}(\breve{K}_N/\breve{K}) \cong \mathbb{F}^\times_{q^{d+1}}$ via the character θ.

□

4. Supercuspidal Part of Hyodo-Kato Cohomology

4.1. **Notations.** Let GD be the group $G \times D^\times$ and let v_{GD} be the map

$$v_{GD} : G \times D^\times \longrightarrow \mathbb{Z};\ (g,b) \mapsto v_K(\det(g)\,\mathrm{Norm}(b)).$$

For $i \in \mathbb{Z}$, let $[GD]_i := v_{GD}^{-1}(i\mathbb{Z})$ and put $[G]_i := G \cap [GD]_i$, $[D]_i := D^\times \cap [GD]_i$. Note that there are inclusions

$$\mathcal{O}_D^\times \longrightarrow [GD]_0;\ b \mapsto (id, b)$$
$$G \longrightarrow [GD]_0;\ g \mapsto (g, \Pi_D^{-\det(g)})$$

but their image does not commute.

Let $\theta : \mathbb{F}^\times_{q^{d+1}} \to \breve{K}$ be a character. We can view θ as a character of $[D]_{d+1}$ via

$$[D]_{d+1} \cong \mathcal{O}_D^\times \varpi^{\mathbb{Z}} \to \mathcal{O}_D^\times \to \mathcal{O}_D^\times/(1+\Pi_D\mathcal{O}_D) \cong \mathbb{F}^\times_{q^{d+1}} \to \breve{K}.$$

It is associated, via Deligne-Lusztig correspondence, a representation $\overline{\pi}_\theta$ of $\mathrm{GL}_{d+1}(\mathbb{F}_q)$. We can view it as a representation of $\mathrm{GL}_{d+1}(\mathcal{O}_K)\varpi^{\mathbb{Z}}$ via $\mathrm{GL}_{d+1}(\mathcal{O}_K) \to \mathrm{GL}_{d+1}(\mathbb{F}_q)$. We consider the following representations via induction:

$$\tilde{\pi}(\theta) := \text{c-ind}^{[G]_{d+1}}_{\mathrm{GL}_{d+1}(\mathcal{O}_K)\varpi^{\mathbb{Z}}} \overline{\pi}_\theta$$
$$\pi(\theta) := \text{c-ind}^{G}_{\mathrm{GL}_{d+1}(\mathcal{O}_K)\varpi^{\mathbb{Z}}} \overline{\pi}_\theta$$
$$\rho(\theta) := \text{c-ind}^{D^\times}_{[D]_{d+1}} \theta.$$

On the Galois side, let $\widetilde{\theta}$ be the character of $[W_K]_{d+1} := I_K\langle\varphi^{d+1}\rangle^{\mathbb{Z}}$ such that $\widetilde{\theta}(\varphi^{d+1}) = (-1)^d q^{d(d+1)/2}$ and $\widetilde{\theta}|_{I_K}$ factors as $I_K \to I_K/I_{K_N} \cong \mathbb{F}^\times_{q^{d+1}} \xrightarrow{\theta} \breve{K}$. Let

$$\sigma(\theta) := \mathrm{ind}^{W_K}_{[W_K]_{d+1}} \widetilde{\theta}$$

be the Weil representation. Let $M(\theta)$ be the $(d+1)$-dimensional (φ, N, G_K)-module over K^{ur} described as the following: $M(\theta)$ admits a basis $\{e_0, ..., e_d\}$ such that $\varphi(e_i) = e_{i+1}$ for $0 \leq i < d$ and $\varphi(e_d) = (-1)^d q^{d(d+1)/2} e_0$, the monodromy acts trivially, the I_K-action factors through $\mathrm{Gal}(\breve{K}_N/\breve{K}) \cong \mathbb{F}_{q^{d+1}}^\times$ via $g(e_i) = \sigma^i(\theta(g))e_i$ and the Frobenius in G_K acts trivially on e_i. So $\sigma(\theta)$ is the Weil-Deligne representation associated to $M(\theta)$.

There is a natural action of GD on $\mathcal{M}^1_{Dr}$ which is non-canonically isomorphic to $\Sigma^1 \times \mathbb{Z}$. Let $\mathcal{M}^{1,\varpi}_{Dr}$ be the quotient $\mathcal{M}^1_{Dr}/\varpi^{\mathbb{Z}}$. It has a formal model over $\mathcal{O}_K$ (the Weil descent datum on $\mathcal{M}^1_{Dr}/\varpi^{\mathbb{Z}}$ is effective cf. [22, 3.49]) which we still denote by $\mathcal{M}^{1,\varpi}_{Dr}$. If we identify Σ^1_K with $\Sigma^1_K = \Sigma^1_K \times \{0\} \subseteq \Sigma^1_K \times \mathbb{Z}/(d+1)\mathbb{Z} \cong \mathcal{M}^{1,\varpi}_{Dr}$, the action of GD induces one on $\mathcal{M}^{1,\varpi}_{Dr}$. The stabilizer of $\Sigma^1_K \times \{0\}$ can be identified with $[GD]_{d+1}$.

4.2. **Restriction to Smooth Locus.** Let J be a finite set of vertices of $\mathcal{BT}$. We define $Y_J := \cap_{s \in J} \mathbb{H}^d_{\mathbb{F}_q, st(s)}$ to be the intersection of irreducible components corresponding to $s \in J$ and $\mathring{Y}_J := Y_J \backslash (\cup_{s \notin J} Y_s)$. Recall that we have the following maps

$$\Sigma^1_K \xrightarrow{\ \pi\ } \Sigma^0_K = \mathbb{H}^d_K \xrightarrow{\ sp\ } \mathbb{H}^d_{\mathbb{F}_q}.$$

Theorem 4.1. *The inclusion of tubes of* Σ^1_C

$$(\pi^{-1}(]\mathring{Y}_J[^\dagger_{\Sigma^0}))_C \longrightarrow (\pi^{-1}(]Y_J[^\dagger_{\Sigma^0}))_C$$

induces an isomorphism of Hyodo-Kato cohomologies

$$H^*_{\mathrm{HK}}(\pi^{-1}(]Y_J[^\dagger_{\Sigma^0}))_C) \xrightarrow{\cong} H^*_{\mathrm{HK}}(\pi^{-1}(]\mathring{Y}_J[^\dagger_{\Sigma^0}))_C).$$

Proof. Take $\mathcal{X} = \mathbb{H}^d_{\mathcal{O}_{\breve{K}}}$, the condition of [16, Thm. 5.1] is satisfied thus we get a bijection

$$H^*_{\mathrm{dR}}(\pi^{-1}(]Y_J[^\dagger_{\Sigma^0})) \xrightarrow{\cong} H^*_{\mathrm{dR}}(\pi^{-1}(]\mathring{Y}_J[^\dagger_{\Sigma^0})). \tag{4.1}$$

In fact, they are finite dimensional K^{ur}-vector spaces. Since $]Y_J[_{\Sigma^0} \cong \mathbb{H}^d_{\breve{K}, st(\sigma)}$ for some simplex σ of $\mathcal{BT}$, use the bijection above, it suffices to see overconvergent de Rham cohomologies of $\pi^{-1}(\mathbb{H}^d_{\breve{K}, \mathring{\sigma}})$ are finite dimensional. We have an explicit description (cf. [16, After Lem. 6.7]):

$$\Sigma^1_{\mathring{\sigma}} \cong (\mathcal{T}_{A_k} \times \mathcal{T}_{C_\sigma})/H$$

where $\mathcal{T}_{A_k} = A_{k,\breve{K}}(u_{A_k}^{1/N})$, $\mathcal{T}_{C_\sigma} = C_{\sigma,\breve{K}}((\pi u_{C_\sigma})^{1/N})$ are μ_N-torsors above $A_{k,\breve{K}}$ and $C_{\sigma,\breve{K}}$ respectively, H is the anti-diagonal of μ_N^2. So it suffices to see finiteness for $\mathcal{T}_{A_k}$ and $\mathcal{T}_{C_\sigma}$. Since T_{C_σ} is quasi-compact and smooth, finiteness follows from [13, Thm. A]. For T_{A_k}, it is [16, Cor. 5.11, Thm. 5.8] applied to the semi-open monomial torus A_k.

Applying geometric Hyodo-Kato isomorphism, we have a commutative diagram

$$\begin{array}{ccc}
H^*_{\mathrm{HK}}(\pi^{-1}(]Y_J[^\dagger_{\Sigma^0}))_C) \otimes_{K^{\mathrm{ur}}} C & \longrightarrow & H^*_{\mathrm{dR}}(\pi^{-1}(]Y_J[^\dagger_{\Sigma^0}))_C) \\
\downarrow & & \downarrow \\
H^*_{\mathrm{HK}}(\pi^{-1}(]\mathring{Y}_J[^\dagger_{\Sigma^0}))_C) \otimes_{K^{\mathrm{ur}}} C & \longrightarrow & H^*_{\mathrm{dR}}(\pi^{-1}(]\mathring{Y}_J[^\dagger_{\Sigma^0}))_C)
\end{array}$$

where two horizontal maps are isomorphisms by geometric Hyodo-Kato isomorphism and right vertical map is an isomorphism by (4.1). Thus the left vertical map is an isomorphism which, use finiteness, implies the desired result.

$\square$

Proposition 4.2. *We have isomorphisms compatible with actions of* φ^{d+1} *and* G_K

$$H^r_{\mathrm{HK},\breve{F},c}(\Sigma^1_C)[\theta] \cong \begin{cases} \bigoplus_{s \in \mathcal{BT}_0} H^s_{\mathrm{HK},\breve{F},c}(\Sigma^1_{s,C})[\theta] & r = d \\ 0 & \textit{others}. \end{cases} \tag{4.2}$$

In particular, as a G-module, $H^d_{\mathrm{HK},\check{F},c}(\Sigma^1_C)[\theta]$ is isomorphic to $\pi(\theta)$.

Proof. By functoriality, the Hyodo-Kato morphism is $G\times\mathbb{F}^\times_{q^{d+1}}$-equivariant. Therefore, $H^j_{\mathrm{HK},\check{F},c}(\Sigma^1_{st(\sigma),C})[\theta]=0$ when σ has nonzero dimension because otherwise a non-zero element will induce a nonzero element in $H^j_{\mathrm{dR},c}(\Sigma^1_{st(\sigma),C})[\theta]$. Especially, the Cech spectral sequence given by the covering provided by $\Sigma^1_{st(\sigma)}$

$$E_1^{-r,s}=\bigoplus_{\sigma\in\mathcal{BT}} H^s_{\mathrm{HK},\check{F},c}(\Sigma^1_{st(\sigma),C})\Rightarrow H^{s-r}_{\mathrm{HK},\check{F},c}(\Sigma^1_C) \tag{4.3}$$

degenerates once we take the θ-isotypical part. Apply Theorem 4.1 and 3.1, we get

$$\bigoplus_{s\in\mathcal{BT}_0} H^s_{\mathrm{HK},\check{F},c}(\Sigma^1_{st(s),C})[\theta]\cong\bigoplus_{s\in\mathcal{BT}_0} H^s_{\mathrm{HK},\check{F},c}(\Sigma^1_{s,C})[\theta]\cong\bigoplus_{s\in\mathcal{BT}_0} H^s_{\mathrm{rig},c}(\mathrm{DL}_{\overline{\mathbb{F}}}/\check{K})[\theta].$$

□

4.3. **Jacquet Langlands and Local Langlands.**

Theorem 4.3. *Let θ be a primitive character of $\mathbb{F}^\times_{q^{d+1}}$. We have an isomorphism of G-modules compatible with action of φ, N and G_K.*

$$\mathrm{Hom}_{D^\times}(\rho(\theta),H^r_{\mathrm{HK},c}(\mathcal{M}^{1,\varpi}_{Dr,C}))\cong\begin{cases}\pi(\theta)\otimes M(\theta) & r=d\\ 0 & \mathit{others}.\end{cases} \tag{4.4}$$

Proof. For the first, we forget the action of φ, N and G_K. As G-modules, we have the following isomorphisms

$$\begin{aligned}\mathrm{Hom}_{D^\times}(\rho(\theta),H^d_{\mathrm{HK},c}(\mathcal{M}^{1,\varpi}_{Dr,C}))&\cong\mathrm{Hom}_{D^\times}(\text{c-ind}^{D^\times}_{[D]_{d+1}}\theta,\text{c-ind}^{GD}_{[GD]_{d+1}}H^d_{\mathrm{HK},c}(\Sigma^1_C))\\&\cong\mathrm{Hom}_{[D]_{d+1}}(\theta,\text{c-ind}^{G}_{[G]_{d+1}}H^d_{\mathrm{HK},c}(\Sigma^1_C))\\&\cong\text{c-ind}^{G}_{[G]_{d+1}}\mathrm{Hom}_{\mathbb{F}^\times_{q^{d+1}}}(\theta,H^d_{\mathrm{HK},c}(\Sigma^1_C))\\&\cong\text{c-ind}^{G}_{[G]_{d+1}}\pi(\theta)|_{[G]_{d+1}}\\&\cong\text{c-ind}^{G}_{[G]_{d+1}}(\text{c-ind}^{G}_{[G]_{d+1}}\widetilde{\pi}(\theta))|_{[G]_{d+1}}\\&\cong\text{c-ind}^{G}_{[G]_{d+1}}\bigoplus_{x\in G/[G]_{d+1}}c_x(\widetilde{\pi}(\theta))\\&\cong\bigoplus_{x\in G/[G]_{d+1}}\text{c-ind}^{G}_{[G]_{d+1}}c_x(\widetilde{\pi}(\theta))\\&\cong\pi(\theta)^{|G/[G]_{d+1}|}=\pi(\theta)^{d+1}.\end{aligned}$$

The second isomorphism is by adjunction, the forth one is proposition 4.2, the sixth one is Mackey decomposition using $[G]_{d+1}$ is a normal subgroup, the others are easy. When $r\neq d$, the same computation together with proposition 4.2 shows the Hom set is zero.

Now we consider the action of φ, N, G_K. From the computation above, we see $\mathrm{Hom}_G(\pi(\theta),H^d_{\mathrm{HK},c}(\mathcal{M}^{1,\varpi}_{Dr,C})[\rho(\theta)])$ is a $(d+1)$-dimensional (φ,N,G_K)-module and the $(d+1)$ copies of $\pi(\theta)$ corresponds to $(d+1)$ connected components of $\mathcal{M}^{0,\varpi}_{Dr,C}$. Consider the period map ξ described in section 2.3.1, the Frobenius action φ translate the connected components by $+1$. On $H^d_{\mathrm{HK},c}(\mathcal{M}^{1,\varpi}_{Dr,C})$, φ^{d+1} acts via multiplication via $(-1)^d q^{d(d+1)/2}$ by proposition 3.8 and 4.2. To see N and G_K act as desired, note that the period map is a local isomorphism, compatible with Weil descent data. Thus it follows by a same argument as in section 3.2.2 and that the Cech spectral sequence (4.3) is N and G_K equivariant.

□

References

1. Henri Carayol, *Nonabelian Lubin-Tate theory*, In Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), volume 11 de Perspect. Math, 15–39.
2. Pierre Colmez and Wieslawa Niziol, *On p-adic comparison theorems for rigid analytic varieties, I.*, Münster Journal of Mathematics **13** (2020), no. 2.
3. Pierre Colmez and Wiesława Nizioł, *On the cohomology of p-adic analytic spaces, I: The basic comparison theorem*, Journal of Algebraic Geometry **34** (2024).
4. J-F Dat, *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Inventiones mathematicae **169** (2007), no. 1, 75–152.
5. Ehud De Shalit, *Residues on buildings and de rham cohomology of p-adic symmetric domains*, Duke Math.J **106** (2001), no. 1, 123–191.
6. Pierre Deligne and George Lusztig, *Representations of reductive groups over finite fields*, Annals of Mathematics **103** (1976), no. 1, 103–161.
7. Gabriel Dospinescu, Pierre Colmez, and Wieslawa Niziol, *Cohomologie p-adique de la tour de drinfeld: le cas de la dimension 1*, Journal of the American Mathematical Society (2019).
8. ______, *Cohomologie des courbes analytiques p-adiques*, Cambridge Journal of Mathematics **10** (2022), no. 3, 511–655.
9. V.G Drinfeld, *Elliptic modules*, Sbornik: Mathematics **23** (1974), no. 4, 561–592.
10. ______, *Coverings of p-adic symmetric regions*, Funktsional'nyi Analiz i ego Prilozheniya **10** (1976), no. 2, 29–40.
11. Jean Fresnel and Marius Van der Put, *Rigid analytic geometry and its applications*, vol. 218, Springer Science & Business Media, 2012.
12. O Goldman and N Iwahori, *The space of p-adic seminorms*, Acta Mathematica **109** (1963), no. 1, 137–177.
13. Elmar Grosse-Klönne, *Finiteness of de Rham cohomology in rigid analysis*, Duke Mathematical Journal **113** (2002), no. 1, 57 – 91.
14. Elmar Grosse-Klönne, *On the Crystalline cohomology of Deligne-Lusztig varieties*, Finite Fields and Their Applications **13** (2007), no. 4, 896–921.
15. Gelbart Harder Iwasawa, Jacquet Katz Piatetski-Shapiro, Raghavan Shintani Stark Zagier, and Nicholas M Katz, *Crystalline cohomology, dieudonné modules, and jacobi sums*, Automorphic Forms, Representation Theory and Arithmetic: Papers presented at the Bombay Colloquium 1979, Springer, 1981, pp. 165–246.
16. Damien Junger, *Cohomologie de de Rham du revêtement modéré de l'espace de Drinfeld*, Journal of the Institute of Mathematics of Jussieu, 1–43.
17. ______, *Équations pour le premier revétement de l'espace symétrique de Drinfeld*, arXiv preprint arXiv:2202.01018 (2022).
18. ______, *Cohomologie analytique des arrangements d'hyperplans*, Algebra & Number Theory **17** (2023), no. 1, 1–43.
19. KIRAN S KEDLAYA, *Finiteness of rigid cohomology with coefficients*, DUKE MATHEMATICAL JOURNAL **134** (2006), no. 1.
20. Wieslawa Niziol, Sally Gilles, and Poitr Achinger, *Compactly supported pro-étale cohomology of analytic varieties*, Preprint 2025.
21. M Rapoport, *On the bad reduction of Shimura varieties. automorphic forms, Shimura varieties, and l-functions, vol. II (ann arbor, mi, 1988), 253–321*, Perspect. Math **11**.
22. M. Rapoport and T. Zink, *Period spaces for p-divisible groups*, Annals of mathematics studies, Princeton University Press, 1996.
23. Haoran Wang, *L'espace symétrique de Drinfeld et correspondance de Langlands locale i*, Mathematische Zeitschrift **278** (2014), no. 3, 829–857.